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ULTRASONIC WAVE PROPAGATION IN DEFORMED ISOTROPIC ELASTIC MATERIALS

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Abstract—Ultrasonic wave propagation in a deformed solid is studied on the basis of the second-order theory of elasticity. The deformation considered is such a kind as one of principal axes of the stress has the same direction at any point and the other two rotate. The fundamental differential equations of the second order which govern ultrasonic wave propagation along the fixed principal direction are reduced to the first order equations by adequate approximation. Then the effects of non-uniformity of the deformation, especially rotation of principal axes of the stress, on waves are examined. It is shown for the quasi-transverse wave that principal axes of the stress are no longer polarization directions but there exist new directions called characteristic directions as in three-dimensional photo-elasticity.

1. INTRODUCTION

ULTRASONIC waves in deformed elastic materials have been studied from two points of view. One is to determine the third-order elastic constants [1] which are required in solid state physics and another is to analyze stress distributions [2–5]. In the former case it is sufficient to study uniform deformations on which ultrasonic waves are superposed, but in the latter we have to treat inhomogeneous deformations. Tokuoka and Iwashimizu [5] showed in two-dimensional stress states the stress–acoustical law holds in certain approximation.

In this paper we discuss the effects of rotation of principal axes of the stress along the wave normal on amplitudes and phases of ultrasonic waves. It is shown that phases of waves are influenced by non-uniformity of deformations and transverse waves do not exhibit such polarization phenomena as in uniform deformations [6, 7] and in undeformed crystals [7, 8] but have new directions called characteristic directions as in three-dimensional photo-elasticity [9].

2. BASIC RELATIONS OF THE THEORY OF AN INFINITESIMAL ELASTIC DEFORMATION SUPERPOSED ON A SMALL ELASTIC DEFORMATION

An elastic material of the Green type is deformed from the natural state to the static state where the stress tensor of Cauchy is t_{ij}. A superposed infinitesimal wave is described by w_k subject to the equation [5]

$$t_{lm}\frac{\partial^2 w_k}{\partial x_l \partial x_m}+\frac{\partial}{\partial x_l}\left[S_{klrs}\frac{\partial w_r}{\partial x_s}\right] = \rho \ddot{w}_k, \tag{2.1}$$

where x_k is rectangular Cartesian coordinate in the deformed state, and the usual summation convention is used. Also

$$\frac{\partial}{\partial x_3}\left[(1+p+qe_{11})\left[\mu\left(\frac{\partial w_k}{\partial x_l}+\frac{\partial w_l}{\partial x_k}\right)+\lambda\frac{\partial w_m}{\partial x_m}\delta_{kl}\right]+2\mu\left[e_{km}\left(\frac{\partial w_m}{\partial x_l}+\frac{\partial w_l}{\partial x_m}\right)+e_{ml}\left(\frac{\partial w_k}{\partial x_m}+\frac{\partial w_m}{\partial x_k}\right)\right]\right.$$

$$+2\lambda\left(e_{kl}\frac{\partial w_m}{\partial x_m}+e_{mn}\frac{\partial w_m}{\partial x_n}\delta_{kl}\right)+2(l-m)e\delta_{kl}\frac{\partial w_m}{\partial x_m}+2m\left(\delta_{kl}e_{mn}\frac{\partial w_m}{\partial x_n}+e_{kl}\frac{\partial w_m}{\partial x_m}\right) \tag{2.2}$$

$$\left.+me\left(\frac{\partial w_k}{\partial x_l}+\frac{\partial w_l}{\partial x_k}\right)+\frac{n}{2}(\delta^{rkm}_{sln}+\delta^{skm}_{rln})e_{mn}\frac{\partial w_r}{\partial x_s}+2\lambda\left(e_{kl}\frac{\partial w_m}{\partial x_m}+e_{mn}\frac{\partial w_m}{\partial x_n}\delta_{kl}\right)\right].$$

where e_{kl} and e are the linear strain and the dilatation in the deformed state, and λ, μ, l, m and n are elastic constants defined by the expression of the strain energy Σ as

$$\Sigma = \frac{1}{2}(\lambda+2\mu)\mathrm{I}_E^2-2\mu\mathrm{II}_E+\frac{l+2m}{3}\mathrm{I}_E^3-2m\mathrm{I}_E\mathrm{II}_E+n\mathrm{III}_E. \tag{2.3}$$

I_E, II_E and III_E being principal invariants of Lagrangian strain.

In [5] (2.1) and (2.2) are derived in the case $l = m = n = 0$.

3. SIMPLIFICATION OF THE FUNDAMENTAL EQUATION (2.1)

At first we assume that the deformation state on which ultrasonic waves are superposed is such as one of principal directions of the stress coincides with the x_3-direction at any point of the material, that is

$$e_{13} = e_{23} = 0, \qquad \frac{\partial t_{33}}{\partial x^3} = 0. \tag{3.1}$$

In this deformation state consider a ultrasonic wave propagating along the x_3-direction. Then

$$\frac{\partial w_k}{\partial x_3} \gg \frac{\partial w_k}{\partial x_1},\ \frac{\partial w_k}{\partial x_2} \qquad (k = 1, 2, 3) \tag{3.2}$$

hold because of smallness of the wavelength. Taking into account (3.2) the equations (2.1) are reduced to

$$
\begin{aligned}
&\frac{\partial}{\partial x_3}\left[(1+p+qe_{11})\frac{\partial w_1}{\partial x_3}+qe_{12}\frac{\partial w_2}{\partial x_3}\right]-\frac{1}{v_{0\perp}^2}(1-e)\frac{\partial^2 w_1}{\partial t^2} = -\left(1+\frac{\lambda}{\mu}\right)\frac{\partial^2 w_1}{\partial x_3\partial x_1}\\
&\qquad -\left[2\frac{\lambda+m}{\mu}\left(\frac{\partial e_{1\alpha}}{\partial x_\alpha}+\frac{\partial e_{33}}{\partial x_1}\right)+2\frac{l-m}{\mu}\frac{\partial e}{\partial x_1}+\frac{n}{\mu}\left(\frac{\partial e_{22}}{\partial x_1}-\frac{\partial e_{21}}{\partial x_2}\right)\right]\frac{\partial w_3}{\partial x_3},\\
&\frac{\partial}{\partial x_3}\left[(1+p+qe_{22})\frac{\partial w_2}{\partial x_3}+qe_{21}\frac{\partial w_1}{\partial x_3}\right]-\frac{1}{v_{0\perp}^2}(1-e)\frac{\partial^2 w_2}{\partial t^2} = -\left(1+\frac{\lambda}{\mu}\right)\frac{\partial^2 w_3}{\partial x_3\partial x_2}\\
&\qquad -\left[2\frac{\lambda+m}{\mu}\left(\frac{\partial e_{2\alpha}}{\partial x_\alpha}+\frac{\partial e_{33}}{\partial x_2}\right)+2\frac{l-m}{\mu}\frac{\partial e}{\partial x_2}+\frac{n}{\mu}\left(\frac{\partial e_{11}}{\partial x_2}-\frac{\partial e_{12}}{\partial x_1}\right)\right]\frac{\partial w_3}{\partial x_3},
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial x_3}\left[(1+r)\frac{\partial w_3}{\partial x_3}\right]-\frac{1}{v_{0\parallel}^2}(1-e)\frac{\partial^2 w_3}{\partial t^2}\\
&\quad = \frac{-1}{\lambda+2\mu}\left[(\lambda+\mu)\frac{\partial^2 w_\alpha}{\partial x_\alpha\partial x_3}+\left(2\mu+\frac{n}{2}\right)\left(\frac{\partial e_{33}}{\partial x_\alpha}\frac{\partial w_\alpha}{\partial x_3}+\frac{\partial e_{\alpha\beta}}{\partial x_\alpha}\frac{\partial w_\beta}{\partial x_\beta}\right)+\left(m-\frac{n}{2}\frac{\partial e}{\partial x_\alpha}\frac{\partial w_\alpha}{\partial x_3}\right)\right],
\end{aligned}
\tag{3.4}
$$

where

$$
\begin{aligned}
p &= \left(\frac{m}{\mu}-\frac{n}{2\mu}-1\right)e+2\left(1+\frac{n}{4\mu}\right)e_{33}+\frac{t_{33}}{\mu},\\
q &= 2\left(1+\frac{n}{4\mu}\right),\\
r &= \left(\frac{2l}{\lambda+2\mu}-1\right)e+4\left(\frac{m}{\lambda+2\mu}+1\right)e_{33}+\frac{t_{33}}{\lambda+2\mu},\\
v_{0\perp}^2 &= \frac{\mu}{\rho_0}, \qquad v_{0\parallel}^2 = \frac{\lambda+2\mu}{\rho_0}
\end{aligned}
\tag{3.5}
$$

and ρ_0 is the density in the undeformed state and Greek indices α and β take the values 1 and 2. In derivation of (3.3) and (3.4) we used (3.1) and retained only terms containing derivatives with respect to x_3.

Because of the right hand members in (3.3) and (3.4) the exact transverse wave ($w_3 = 0$) and the exact longitudinal wave ($w_1 = w_2 = 0$) cannot propagate. For example even if the initial and the boundary conditions are satisfied by the transverse wave only, the longitudinal wave may be generated through (3.3) and (3.4) and vice versa. However if we treat two cases (a) incidence of the transverse wave and (b) incidence of the longitudinal wave separately we can simplify (3.3) and (3.4) and show that the quasi-transverse and the quasi-longitudinal waves can propagate without each other.

Case (a): *Incidence of the transverse wave*

In this case w_1 and w_2 are primary and the magnitude of the associated w_3 can be expected to satisfy

$$|w_3| \sim |e_{kl}w_1|, |e_{kl}w_2| \tag{3.6}$$

by (3.4) and (3.2). Then the magnitudes of the right hand members of (3.3) are at most

$$\frac{\partial e_{ij}}{\partial x_\beta}\,\frac{\partial}{\partial x_3}(e_{kl}w_\alpha)\,,$$

while those of the left hand members of (3.3) are at least

$$\frac{\partial e_{kl}}{\partial x_3}\,\frac{\partial w_\alpha}{\partial x_3}\qquad (\alpha, \beta = 1, 2).$$

Therefore assuming

$$|e_{kl}| \sim \left|\frac{\partial e_{kl}}{\partial x_i}\right|,$$

we can neglect the right hand members of (3.3), and have

$$\begin{aligned} &\frac{\partial}{\partial x_3}\left[(1+p+qe_{11})\frac{\partial w_1}{\partial x_3}+qe_{12}\frac{\partial w_2}{\partial x_3}\right]-\frac{1}{v_{0\perp}^2}(1-e)\frac{\partial^2 w_1}{\partial t^2}=0 \\ &\frac{\partial}{\partial x_3}\left[qe_{21}\frac{\partial w_1}{\partial x_3}+(1+p+qe_{22})\frac{\partial w_2}{\partial x_3}\right]-\frac{1}{v_{0\perp}^2}(1-e)\frac{\partial^2 w_2}{\partial t^2}=0. \end{aligned} \tag{3.7}$$

We express the solutions of (3.7) as

$$w_\alpha = \overline{W}_\alpha \exp[i(\omega t - k_{0\perp}x_3)] \qquad (\alpha = 1, 2). \tag{3.8}$$

where $\overline{W}_1$ and $\overline{W}_2$ are functions of x_1, x_2 and x_3, and $k_{0\perp} = \omega/v_{0\perp} \gg 1$. Using (3.8) and (3.7), and taking into account $k_{0\perp} \gg 1$ and the relation expected in advance that $\overline{W}_\alpha$ and their derivatives are of the same order magnitude, we have

$$\begin{aligned} \frac{\mathrm{d}\overline{W}_1}{\mathrm{d}x_3} &= \frac{i}{2}k_{0\perp}(p+e+qe_{11})\overline{W}_1+\frac{i}{2}k_{0\perp}qe_{12}\overline{W}_2, \\ \frac{\mathrm{d}\overline{W}_2}{\mathrm{d}x_3} &= \frac{i}{2}k_{0\perp}qe_{21}\overline{W}_1+\frac{i}{2}k_{0\perp}(p+e+qe_{22})\overline{W}_2. \end{aligned} \tag{3.9}$$

Furthermore we use the transformation

$$\overline{W}_\alpha = W_\alpha \exp\left[\frac{i}{2}k_{0\perp}\int\left\{p+e-\frac{1}{2}q(e_{11}+e_{22})\right\}\mathrm{d}x_3\right] (\alpha = 1, 2) \tag{3.10}$$

to simplify (3.9), and the results are

$$\begin{aligned} \frac{\mathrm{d}W_1}{\mathrm{d}x_3} &= \frac{1}{2}iC(e_{11}-e_{22})W_1+iCe_{12}W_2, \\ \frac{\mathrm{d}W_2}{\mathrm{d}x_3} &= iCe_{21}W_1-\frac{1}{2}iC(e_{11}-e_{22})W_2, \end{aligned} \tag{3.11}$$

where $C = \frac{1}{2}qk_{0\perp}$. Equations (3.11) are the final equations governing the quasi-transverse wave in the case (a).

For later use we transform (3.11) to local principal axes of the stress. The components of W_α and $e_{\alpha\beta}$ ($\alpha, \beta = 1, 2$) in these axes are denoted by W'_α and e_α ($\alpha = 1, 2$) respectively and their transformations are

$$\begin{aligned} W_1 &= W'_1 \cos\theta - W'_2 \sin\theta, \\ W_2 &= W'_1 \sin\theta + W'_2 \cos\theta, \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} e_{11} &= \tfrac{1}{2}(e_1+e_2)+\tfrac{1}{2}(e_1-e_2)\cos 2\theta, \\ e_{22} &= \tfrac{1}{2}(e_1+e_2)-\tfrac{1}{2}(e_1-e_2)\cos 2\theta, \\ e_{12} &= e_{21} = \tfrac{1}{2}(e_1-e_2)\sin 2\theta, \end{aligned} \tag{3.13}$$

where θ is the angle between the principal axis and the x_3-axis and is a function of x_3. If we put $\theta = 0$ after using (3.12) and (3.13) to (3.11) and carrying out indicated differentiation, we obtain the required equations;

$$\begin{aligned} \frac{dW'_1}{dx_3} &= \frac{1}{2}iC(e_1-e_2)W'_1 + \frac{d\theta}{dx_3}W'_2, \\ \frac{dW'_2}{dx_3} &= -\frac{d\theta}{dx_3}W'_1 - \frac{1}{2}iC(e_1-e_2)W'_2. \end{aligned} \tag{3.14}$$

Case (b): *Incidence of the quasi-longitudinal wave*

In this case w_3 is primary and the magnitude of the associated w_1 and w_2 can also be expected to satisfy

$$|w_1|, |w_2| \sim |e_{kl}w_3|$$

by (3.3) and (3.2). Then as in the case (a) (3.4) can be replaced by

$$\frac{\partial}{\partial x_3}\left[(1+r)\frac{\partial w_3}{\partial x_3}\right] - \frac{1}{v_{0\parallel}^2}(1-e)\frac{\partial^2 w_3}{\partial t^2} = 0. \tag{3.15}$$

In the same way as in the case (a) we can simplify (3.15) as

$$\frac{dW_3}{dx_3} = ik_{0\parallel}\frac{r-e}{2}W_3, \tag{3.16}$$

where

$$w_3 = W_3 \exp\left\{\frac{i}{2}k_{0\parallel}\int_{x_{30}}^{x_3}(r-e)\,dx_3\right\}. \tag{3.17}$$

Then

$$W_3 = D\exp\left\{\frac{i}{2}k_{0\parallel}\int_{x_{30}}^{x_3}(r-e)\,dx_3\right\}, \tag{3.18}$$

where D is a constant.

Therefore the effect of the deformation on the quasi-longitudinal wave is expressed by the additional phase δ;

$$\delta = \tfrac{1}{2}k_0 \int_{x_{30}}^{x_3} (r-e)\,\mathrm{d}x_3. \tag{3.19}$$

4. BEHAVIORS OF $e_1 - e_2$ AND $\mathrm{d}\theta/\mathrm{d}x_3$

Since we confine the type of the deformation by the assumption (3.1), functions in coefficients of (3.11) and (3.14) are also restricted. Here we consider behaviors of these functions and use these results in examining (3.11) or (3.14).

By almost the same discussions in [10] the restriction (3.1) gives following general forms of stress components:

$$\begin{aligned} t_{11} &= \frac{\partial^2\chi_0}{\partial x_2^2} + x_3\frac{\partial^2\chi_1}{\partial x_2^2} - \frac{1}{2}\frac{\sigma}{1+\sigma}x_3^2\frac{\partial^2\Theta_0}{\partial x_2^2}, \\ t_{22} &= \frac{\partial^2\chi_0}{\partial x_1^2} + x_3\frac{\partial^2\chi_1}{\partial x_1^2} - \frac{1}{2}\frac{\sigma}{1+\sigma}x_3^2\frac{\partial^2\Theta_0}{\partial x_1^2}, \\ t_{12} &= -\frac{\partial^2\chi_0}{\partial x_1\partial x_2} - x_3\frac{\partial^2\chi_1}{\partial x_1\partial x_2} + \frac{1}{2}\frac{\sigma}{1+\sigma}x_3^2\frac{\partial^2\Theta_0}{\partial x_1\partial x_2}, \end{aligned} \tag{4.1}$$

where χ_0, χ_1 and Θ_0 are functions of x_1 and x_2 only, and satisfy

$$\nabla^2\chi_0 = \Theta_0, \qquad \nabla^2\chi_1 = \text{const.} \tag{4.2}$$

For our purpose it is sufficient to know how coefficient functions in (3.11) and (3.14) depend upon x_3. Therefore we have from (4.1)

$$\begin{aligned} e_{11} - e_{22} &= a_0 + a_1x_3 + a_2x_3^2, \\ e_{12} &= c_0 + c_1x_3 + c_2x_3^2. \end{aligned} \tag{4.3}$$

$$\begin{aligned} \frac{\mathrm{d}\theta}{\mathrm{d}x_3} &= \frac{(c_1a_0 - c_0a_1) + 2(c_2a_0 - c_0a_2)x_3 + (c_2a_1 - c_1a_2)x_3^2}{(a_0 + a_1x_3 + a_2x_3^2)^2 + 4(c_0 + c_1x_3 + c_2x_3^2)^2}, \\ e_1 - e_2 &= [(a_0 + a_1x_3 + a_2x_3^2)^2 + 4(c_0 + c_1x_3 + c_2x_3^2)^2]^{\frac{1}{2}}, \end{aligned} \tag{4.4}$$

where a's and c's are functions of x_1 and x_2 only.

Let us define the quantity κ by

$$\kappa = \frac{\frac{1}{2}C(e_1 - e_2)}{\dfrac{\mathrm{d}\theta}{\mathrm{d}x_3}}. \tag{4.5}$$

Then by (4.4)

$$\kappa = \frac{1}{2}C\frac{[(a_0 + a_1x_3 + a_2x_3^2)^2 + 4(c_0 + c_1x_3 + c_2x_3^2)^2]^{3/2}}{(c_1a_0 - c_0a_1) + 2(c_2a_0 - c_0a_2)x_3 + (c_2a_1 - c_1a_2)x_3^2}. \tag{4.6}$$

We can show that the case $|\kappa| \gg 1$ and the case $e_1 - e_2$ and $d\theta/dx_3$, therefore κ are nearly constant for certain interval of x_3 are possible but the case $|\kappa| \ll 1$ is impossible. As an example we show in Fig. 1 the relation between κ and x_3 for one possible case

$$a_0 = -2c_0, \qquad a_1 = 0{\cdot}4c_0, \qquad c_1 = 0{\cdot}2c_0, \qquad a_2 = c_2 = 0.$$

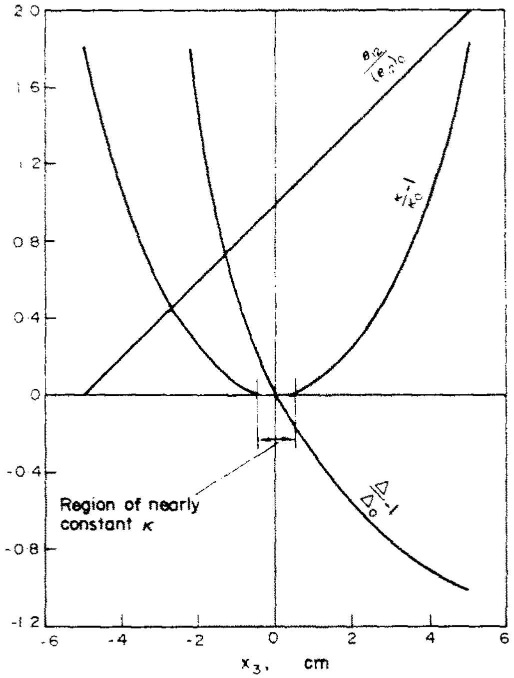

FIG. 1. Behaviors of κ, Δ and e_{12}.

Behaviors of $\Delta = (e_{11} - e_{22}/2e_{12})$ and e_{12} are also shown in the same figure, where κ_0, Δ_0 and $(e_{12})_0$ are values of κ, Δ and e_{12} at $x_3 = 0$, respectively. It is seen from this that the variation of κ in the interval $|x_3| \leq 0{\cdot}5$ is less than 1 per cent, while that of Δ is much larger.

We consider the characters of the quasi-transverse wave in two cases $(a_1)|\kappa| \gg 1$ and (a_2) constant $e_1 - e_2$ and $d\theta/dx_3$ as special cases.

5. TWO SPECIAL CASES

Case (a_1): $|\kappa| \gg 1$

This corresponds to the case where rotation of principal axes is very small. Then (3.14) become

$$\frac{dW'_1}{dx_3} = \frac{1}{2} iC(e_1 - e_2)W'_1,$$
$$\frac{dW'_2}{dx_3} = -\frac{1}{2} iC(e_1 - e_2)W'_2, \tag{5.1}$$

by neglecting terms containing $\mathrm{d}\theta/\mathrm{d}x_3$. (5.1) show that quasi-transverse wave is polarized in two slightly rotating principal axes of the stress, and the relative phase difference of these polarized waves δ is given by

$$\delta = C\int_{x_{30}}^{x_3} (e_1 - e_2)\,\mathrm{d}x_3. \tag{5.2}$$

Case (a_2): $e_1 - e_2$ *and* $\mathrm{d}\theta/\mathrm{d}x_3$ *are constant*

For brevity we put

$$\tfrac{1}{2}C(e_1 - e_2) = E, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}x_3} = F. \tag{5.3}$$

Then (3.14) has the following fundamental solution matrix whose two columns are two linearly independent solutions:

$$\begin{matrix} \cos Px_3 + \dfrac{iE}{P}\sin Px_3, & \dfrac{F}{P}\sin Px_3 \\ -\dfrac{F}{P}\sin Px_3, & \cos Px_3 - \dfrac{iE}{P}\sin Px_3 \end{matrix} \tag{5.4}$$

where $P = \sqrt{E^2 + F^2}$.

We consider an interesting case in which one of the components in the principal axes of the incident wave, say W_2', is zero at $x_3 = 0$. Then the solutions are

$$\begin{aligned} W_1' &= D_1(1 - K^2)^{\frac{1}{2}}\exp(i\Theta_1), \\ W_2' &= -D_1 K, \end{aligned} \tag{5.5}$$

where D_1 is a constant and

$$\begin{aligned} K &= \frac{F}{P}\sin Px_3, \\ \Theta_1 &= \tan^{-1}\left(\frac{E}{P}\tan Px_3\right). \end{aligned} \tag{5.6}$$

Therefore even if only one component W_1' is present at $x_3 = 0$, another component W_2' is generated during propagation and these two have the relative phase difference $\delta = \Theta_1 - n\pi$ (n: interger). When we detect these waves at $x_3 = x_3$ by a Y-cut quartz crystal with its axis making an angle ψ with the principal axis, the observed quantity is the amplitude of the displacement component w given by

$$w = w_1'\cos\psi - w_2'\sin\psi, \tag{5.7}$$

where w_1' and w_2' are displacement components referred to principal axes:

$$\begin{aligned} w_1' &= D_1(1 - K^2)^{\frac{1}{2}}\cos\left[\omega t - k_{0_2}\left\{x_3 - \frac{1}{2}\int\left[p + e + \frac{1}{4}q(e_1 - e_2)\right]\mathrm{d}x_3\right\} - \Theta_1\right], \\ w_2' &= -D_1 K\cos\left[\omega t - k_{0_2}\left\{x_3 + \frac{1}{2}\int\left[p - e - \frac{1}{4}q(e_1 + e_2)\right]\mathrm{d}x_3\right\}\right]. \end{aligned} \tag{5.8}$$

(5.8) are obtained from (5.5), (3.10) and (3.8). (5.7) can be rewritten as

$$w = D_1 r \cos(\omega t - \zeta), \tag{5.9}$$

where the observed amplitude $D_1 r$ is given by

$$r^2 = \tfrac{1}{2} + \tfrac{1}{2}(1-2K^2)\cos 2\psi - K(1-K^2)^{\frac{1}{2}}\cos\Theta_1 \sin 2\psi. \tag{5.10}$$

We can determine E and F by using (5.10) and rotating the receiving quartz crystal. When $|K| \ll 1$, $r \simeq \frac{1}{2}(1+\cos 2\psi)$, which corresponds to the case (a_1).

Figures 2 and 3 show the relations between r^2 and x_3 and between Θ_1 and x_3 for several values of E, F and ψ.

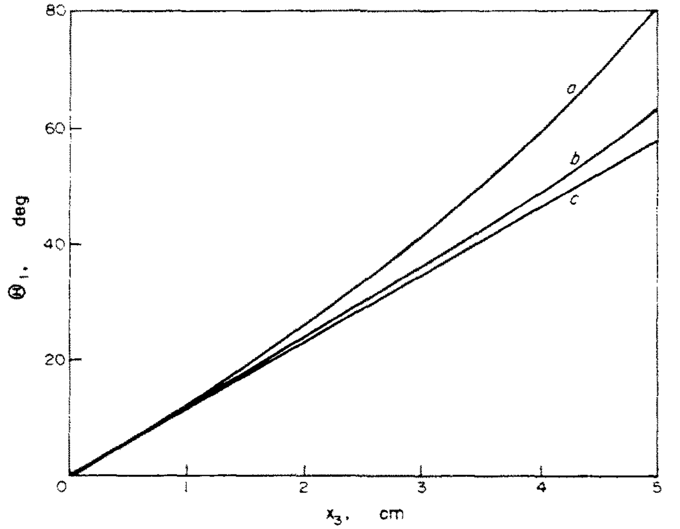

FIG. 2. $\Theta_1 - x_3$ relation: (a) $E = F = \frac{1}{5}$(1/cm); (b) $E = \frac{1}{5}$, $F = \frac{1}{10}$(1/cm); (c) $E = \frac{1}{5}$, $F = 0$(1/cm).

6. GENERAL PROPERTIES OF TRANSVERSE WAVES—CHARACTERISTIC DIRECTIONS

Here we discuss the corresponding directions to those treated by H. K. Aben [9] in photoelasticity.

Since the matrix A composed of coefficients in the right hand members of (3.11) or (3.14) has the property $A^* = -A$, where A^* is the complex conjugate transposed matrix of A, the fundamental solution matrix Φ of (3.11) or (3.15) satisfies [12]

$$\Phi(x_3)\Phi^*(x_3) = \text{const. matrix}.$$

Then the solution of (3.14) which is W_0 at $x_3 = x_{30}$ can be expressed as

$$W = U(x_3, x_{30})W_0, \tag{6.1}$$

where $U(x_3, x_{30})$ is certain two by two unitary matrix given by

$$U(x_3, x_{30}) = \Phi(x_3)\Phi^{-1}(x_{30}), \tag{6.2}$$

and

$$W = \begin{Vmatrix} W_1 \\ W_2 \end{Vmatrix}, \qquad W_0 = \begin{Vmatrix} W_{01} \\ W_{02} \end{Vmatrix}.$$

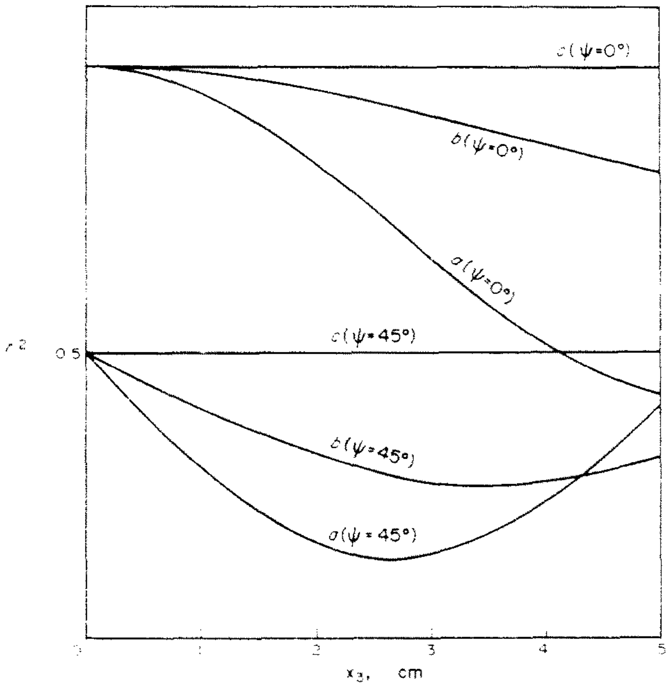

FIG. 3. $r^2 - x_3$ relation: (a) $E = F = \frac{1}{5}(1\text{ cm})$; (b) $E = \frac{1}{5}$, $F = \frac{1}{50}(1\text{ cm})$; (c) $E = \frac{1}{5}$, $F = 0(1\text{ cm})$

It is known [13] that the most general form of a unitary matrix is

$$U(x_3, x_{30}) = \begin{pmatrix} e^{-i(\alpha/2)} & 0 \\ 0 & e^{i(\alpha/2)} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & -i\sin\frac{\beta}{2} \\ -i\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i(\gamma/2)} & 0 \\ 0 & e^{i(\gamma/2)} \end{pmatrix} e^{i(\omega/2)} \tag{6.3}$$

and can also be rewritten as

$$U(x_3, x_{30}) = \begin{pmatrix} \cos\chi_2 & -\sin\chi_2 \\ \sin\chi_2 & \cos\chi_2 \end{pmatrix} \begin{pmatrix} e^{i\xi} & 0 \\ 0 & e^{-i\xi} \end{pmatrix} \begin{pmatrix} \cos\chi_1 & \sin\chi_1 \\ -\sin\chi_1 & \cos\chi_1 \end{pmatrix} e^{i(\omega/2)} \tag{6.4}$$

by using the Eulerian theorem about rotations of a rigid body and the homomorphism of the rotation group with the unitary transformation group with unity determinant. In (6.3) and (6.4) α, β, γ, ω, χ_1, χ_2 and ξ are all real quantities depending on the deformation between x_{30} and x_3. Replacing (6.4) into (6.1) shows that if we refer the result to rotated axes through the angle χ_1 at x_{30} and through the angle χ_2 at x_3 we have

$$\begin{pmatrix} \tilde{W}_1 \\ \tilde{W}_2 \end{pmatrix} = e^{i(\omega/2)} \begin{pmatrix} e^{i\xi} & 0 \\ 0 & e^{-i\xi} \end{pmatrix} \begin{pmatrix} \tilde{W}_{01} \\ \tilde{W}_{02} \end{pmatrix}, \tag{6.5}$$

where we used ~ to denote components in both rotated axes. We call directions determined by angles χ_1 and χ_2 *primary and secondary characteristic directions* after H. K. Aben [9]. From (6.5) the wave linearly polarized in one of the primary characteristic directions is also linearly polarized in the corresponding secondary characteristic direction. This property may be called briefly *two points polarization*. In general these directions do not coincide with principal axes of the stress and can be determined experimentally by rotating two quartz crystals in the transmitted method.

7. CONCLUSIONS

The effects of inhomogeneity of deformations along the propagation direction especially those of rotation of principal axes of the stress are studied. Fundamental equation (3.11) or (3.14) and (3.16) are obtained under the assumptions (3.1) by the adequate approximation.

For a quasi-transverse wave quantities $e_1 - e_2$ and $d\theta/dx_3$ in the coefficients of (3.11) or (3.14) are considered at first and it is shown that the case $|e_1 - e_2| \ll |d\theta/dx_3|$ is impossible. Consideration of two possible interesting cases give the following results;

(a_1) When $|e_1 - e_2| \gg |d\theta/dx_3|$, the wave is polarized in principal axes of the stress and two linearly polarized waves have the relative phase difference (5.2).

(a_2) When $e_1 - e_2$ and $d\theta/dx_3$ are constant, two components in the principal axes are coupled as shown by (5.4).

In general the principal axes are not polarization directions, but it is shown from general properties of solutions of (3.14) or (3.11) that there exist characteristic directions which characterize two points polarization at the incident and the emergent points.

For a quasi-longitudinal wave only its phase is affected by a varying deformation as shown (3.19).

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Абстракт—На основе теории упругости второго порядка, исследуется распространение сверхзвуковой волны в деформированном твердом теле. Рассматриваемая деформация такова, что одна из главных осей напряжений имеет такое же самое направление для произвольной точки, а две другие вращаются. Основные дифференциальные уравнения второго порядка, описывающие распространение сверхзвуковой волны, вдоль неизменяемого главного направления, преобразовываются в уравнения первого порядка, путем соответствующей аппроксимации. Затем, исследуются эффекты неоднородности деформации, а специально вращение главных осей направлений. Оказывается, что для случая квазипоперечной волны главные оси напряжений не являются в дальнейшем направлениями поляризации, но в этом случае существуют новые направления, названные характеристическими, как в трехмерной фотоупругости.